Oregon Invitational Mathematics Tournament
Algebra 2 level exam
May 16, 2015

Name: ________________________________

School: ______________________________

Would you like us to post your score online with your first initial and last name? (please select one): yes no

Instructions:
There are two parts to this exam.
• Part I consists of 15 problems with numerical answers. Each is worth 4 points. Only your final answer will be graded. There will be NO partial credit given. You must write your answer in the place provided.
• Part II consists of 6 problems. Each is worth 10 points. Partial credit may be given for these questions. Each question requires a fully written-up solution. You should clearly write out each of your steps and provide justification when necessary. A correct answer without explanation will receive no credit.
Calculators are not allowed.

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PART I: 4-Point Problems:

(1) To convert from degrees Celsius to Fahrenheit, you multiply by 9/5 and then add 32. If $T$ degrees Celsius is the same as $T$ degrees Fahrenheit, what is the value of $T$?

Answer:

(2) The product of the ages of two friends is 1140. If they are both less than 40 years of age, how old are they?

Answer:
(3) The sum of two numbers is 165 and their product is 2015. Find the sum of their squares.

Answer:

(4) To number pages in a book 1000 digits were used. How many pages are in the book if we start numbering from page three and number it as page 3?

Answer:
(5) Find all real solutions of the equation \((x^2 + 3)^2 - 4x^2 - 24 = 0\).

Answer:

(6) Find the number of integers in 1, 2, \cdots, 300 that are divisible by 3, 5, or 7.

Answer:
(7) For what value(s) of $c$ will the equation \( \frac{1}{x(x - c)} = \frac{1}{1 - c} \) have exactly one solution?

Answer:

(8) Given \( f(x) = \frac{x + 1}{x + 2} \), find the $x$ value in terms of $a$ for which \( f(x) = 1 - \frac{1}{a} \).

Answer:
(9) Find the point \((a, b)\) on the line \(y = 2x - 1\) which is closest to the point \((0, 1)\).

Answer:

(10) A polynomial \(P(x) = x^3 + ax^2 + bx + c\) satisfies \(P(1) = 1^2\), \(P(2) = 2^2\) and \(P(3) = 3^2\). What is \(P(4)\)?

Answer:
(11) A new operation $\otimes$ satisfies the following equations.

- $0 \otimes b = b + 1$
- $(a + 1) \otimes b = a \otimes (a \otimes b)$ for all $a \geq 0$.

What is the value of $4 \otimes 5$?

Answer:

(12) Find the smallest 4 digit number $abcd$ (for example 2375), where $a, b, c, d$ are all distinct integers between 1 to 9 (inclusive) such that the two digit numbers $ab, ac, ad, bc$ and $cd$ are all prime.

Answer:
(13) It takes Tim 4 hours to get from Seattle to Eugene. It takes Mary 6 hours to get from Seattle to Eugene. Aaron’s speed is an average of Tim’s and Mary’s speeds. How long will it take Aaron to get from Seattle to Eugene?

Answer:

(14) Simplify:
\[
\frac{p^3 - 3}{z^2 + 3p} \div \frac{12}{z^2 - p^2} \cdot \frac{3}{z^2 - p^2}
\]
(15) Numbers $a$, $b$, and $c$ are not equal to each other and $a + \frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{a}$. Find $a^2 b^2 c^2$.

Answer:
PART II: 10-Point Problems:

(1) For any numbers $x$ and $y$ such that $x + y \neq 0$, define $x \oplus y$ as

$$x \oplus y = \frac{xy}{x + y}.$$ 

Show that

$$(x \oplus y) \oplus z = x \oplus (y \oplus z)$$

whenever both the expressions make sense.
(2) We know that for any numbers $a$ and $b$, $f(a + b) + f(a - b) = 2f(a) + 2f(b)$.
Is it true that $f(x)$ is even (i.e. $f(x) = f(-x)$ for all values of $x$ from the domain of $f(x)$)?
(3) For which real values of the parameter $t$ does the equation
\[ t^2 x^2 + 2(t^2 - 2)x + 1 = 0 \]
have a unique real solution $x$?
(4) A function from the non-negative integers to the non-negative integers satisfies

- \( f(mn) = mf(n) + nf(m) \)
- \( f(15) = 60 \)
- \( f(10) = 15 \)
- \( f(12) = 12 \)

Find \( f(9) \).
(5) Prove that for any integers $a_1, a_2, \cdots, a_7$, there exist a pair of them $a_i$ and $a_j$ ($1 \leq i, j \leq 7$) such that $a_i + a_j$ or $a_i - a_j$ is divisible by 10.
(6) Prove the following for any $a, b > 0$:

$$\frac{a + b}{2} \geq \sqrt{ab} \geq \frac{2}{\frac{1}{a} + \frac{1}{b}}.$$